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## DIVERGENCE IN DEFORMATION SPACES OF KLEINIAN GROUPS

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This note is based on my talk given in the RIMS on the 4th of December, 2006.

The attention of specialists in the Kleinian group theory is now shifted to the study of the topological structure of deformation spaces after the major problems like Marden's tameness conjecture and the ending lamination conjecture are solved. Although we know, by the resolution of the Bers-Thurston density conjecture ([4]) using the proof of the ending lamination conjecture by Minsky with his collaborators that every finitely generated Kleinian group is an algebraic limit of quasi-conformal deformations of a (minimally parabolic) geometrically finite group, the structure of deformation spaces as topological spaces is far from completely understood.

To understand such a global structure of deformation spaces, the first step would be to give a criterion for sequences in the deformation space to converge or diverge. Let us put it in more concrete terms focusing only on the case of Kleinian groups isomorphic to surface groups. Consider a hyperbolic surface  $S$  of finite type and the space of faithful discrete representations of  $\pi_1(S)$  to  $\mathrm{PSL}_2\mathbb{C}$  preserving the parabolicity modulo conjugacy (both as elements of  $\mathrm{PSL}_2\mathbb{C}$  and complex conjugation), which is usually denoted by  $AH(S)$ . Since the hyperbolic metric of  $S$  determines a Fuchsian representation of  $\pi_1(S)$  to  $\mathrm{PSL}_2\mathbb{R} \subset \mathrm{PSL}_2\mathbb{C}$ , as the space of quasi-conformal deformations of this representation, we can consider the space of quasi-Fuchsian representations  $QF(S)$  embedded as an open set in  $AH(S)$ . What we are interested in is the problem to determine in which directions  $QF(S)$  has frontier in  $AH(S)$  and in which directions it is open-ended. Since by the theory of Ahlfors-Bers,  $QF(S)$  is parametrised by  $\mathcal{T}(S) \times \mathcal{T}(\bar{S})$ , we can describe the directions in  $QF(S)$  in terms of the Teichmüller spaces.

The main results in this talk is the following.

**Theorem 1.** *Let  $\{(m_i, n_i)\}$  be a sequence in  $\mathcal{T}(S) \times \mathcal{T}(\bar{S})$  satisfying the following conditions.*

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- (1)  $\{m_i\}$  converges to a projective lamination  $[\mu^-] \in \mathcal{PML}(S)$  whereas  $\{n_i\}$  converges to  $[\mu^+] \in \mathcal{PML}(S)$ .
- (2) The supports of  $\mu^-$  and  $\mu^+$  share a component  $\mu_0$  which is not a simple closed curve.

Then the sequence  $\{qf(m_i, n_i)\} \subset QF(S)$  diverges in  $AH(S)$ .

**Theorem 2.** Let  $\mu^-$  and  $\mu^+$  be two measured laminations on  $S$  such that the components shared by  $|\mu^-|$  and  $|\mu^+|$  are all simple closed curves, which we denote by  $c_1, \dots, c_r$ .

- (1) Suppose that none of  $c_1, \dots, c_r$  lie on the boundary of supporting surfaces of components of  $\mu^-$  or  $\mu^+$ . Then there is a sequence  $\{(m_i, n_i)\}$  in  $\mathcal{T}(S) \times \mathcal{T}(\bar{S})$  with convergent  $qf(m_i, n_i)$  such that  $m_i$  converges to  $[\bar{\mu}^-]$  and  $n_i$  converges to  $[\bar{\mu}^+]$  and  $|\bar{\mu}^-| = |\mu^-|$ ,  $|\bar{\mu}^+| = |\mu^+|$ . Moreover, if  $|\mu^+| = c_1 \cup \dots \cup c_r$ , we choose  $\{(m_i, n_i)\}$  so that  $qf(m_i, n_i)$  converges exotically to a b-group.
- (2) Otherwise for every  $\{m_i\}$  converging to  $[\mu^-]$  and  $\{n_i\}$  converging to  $[\mu^+]$ , the sequence  $\{qf(m_i, n_i)\} \subset QF(S)$  diverges in  $AH(S)$ .

The proofs of Theorem 1 and Theorem 2 take quite different strategies. For Theorem 1, which is apparently the more complicated case of the two, we can use a rather standard technique of pleated surfaces originally due to Thurston. For Theorem 2, we need to invoke much more sophisticated tool of model manifolds due to Minsky.

In this note we only explain Theorem 1.

### 1. A SKETCH OF PROOF OF THEOREM 1.

Let  $S$  be a hyperbolic surface of finite area. Let  $\phi_i : \pi_1(S) \rightarrow \mathrm{PSL}_2\mathbb{C}$  be a quasi-Fuchsian representation representing  $qf(m_i, n_i)$  as was given in Theorem 1. Let  $G_i$  be the image of  $\phi_i$ , and  $M_i$  the hyperbolic 3-manifold  $\mathbb{H}^3/G_i$ . Since  $G_i$  is a quasi-conformal deformation of the Fuchsian representation of  $\pi_1(S)$  associated to the hyperbolic metric on  $S$ , there is a natural homeomorphism  $\Phi_i : S \times \mathbb{R} \rightarrow M_i$  induced by a quasi-conformal homeomorphism, where we regard  $S \times \mathbb{R}$  as the hyperbolic 3-manifold containing the hyperbolic surface  $S$  in the form of  $S \times \{0\}$  as a totally geodesic submanifold. Since  $G_i$  is quasi-Fuchsian, the manifold  $M_i$  is geometrically finite and has convex core  $C(M_i)$ , which is homeomorphic to  $S \times I$  preserving the parabolicity. We can isotope  $\Phi_i$  above so that  $\Phi_i(S \times [-1, 1]) = C(M_i)$ .

Let  $\Sigma_i^-, \Sigma_i^+$  be the two frontier components of  $C(M_i)$  corresponding to  $\Phi_i(S \times \{-1\})$  and  $\Phi_i(S \times \{1\})$  respectively. The hyperbolic metric on  $M_i$  induces hyperbolic structures on  $\Sigma_i^-$  and  $\Sigma_i^+$  as length metrics.

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We give markings on  $\Sigma_i^-$  and  $\Sigma_i^+$  by natural homeomorphism between  $S$  and  $S \times \{-1\}$  and  $S \times \{1\}$  obtained by forgetting the second coordinates. It should be noted the orientation given on  $\Sigma^+$  is different from the ordinary one induced from  $C(M_i)$ . Let  $(p_i, q_i)$  be points in  $\mathcal{T}(S)$  determined by these hyperbolic structures on  $\Sigma_i^-, \Sigma_i^+$  and markings. Since  $(G_i, \phi_i) = qf(m_i, n_i)$  with respect to the Ahlfors-Bers parametrisation, by Bers' inequality, there is a universal bound  $K$  between the Teichmüller distances between  $m_i, p_i$  and  $n_i, q_i$ .

The pleating loci on  $\Sigma_i^-$  and  $\Sigma_i^+$  give two measured laminations  $\lambda_i^-, \lambda_i^+$  on  $S$  by pulling back them to  $S$  using the inverse of  $\Phi_i|_{S \times \{\pm 1\}}$ . By passing to a subsequence, we can assume that both  $[\lambda_i^-]$  and  $[\lambda_i^+]$  converge to projective laminations  $[\lambda_\infty^-]$  and  $[\lambda_\infty^+]$ . We can also assume that the sequences of supports  $\{|\lambda_i^-|\}$  and  $\{|\lambda_i^+|\}$  converge to geodesic laminations  $\ell_\infty^-$  and  $\ell_\infty^+$  in the Hausdorff topology.

We shall prove Theorem 1 by contradiction. Assume that  $\{(G_i, [\phi_i])\}$  converges to  $(\Gamma, \psi)$  in  $AH(S)$  by taking conjugates and a subsequence. We divide our argument into three cases:

- (1) The first case is when either  $i(\mu^-, \lambda_\infty^-)$  or  $i(\mu^+, \lambda_\infty^+)$  is non-zero.
- (2) The second case is when both  $\lambda_\infty^-$  and  $\lambda_\infty^+$  contain a component shared by  $\mu^\mp$  which is not a simple closed curve.
- (3) Finally, the third case is when either  $\lambda_\infty^+$  or  $\lambda_\infty^-$  is disjoint from any component of  $\mu^+$  shared with  $\mu^-$  that is not a simple closed curve.

In the first case, we assume that  $i(\mu^-, \lambda_\infty^-) > 0$ . The argument for the case when  $i(\mu^+, \lambda_\infty^+) > 0$  is completely the same. By the definition of the Thurston compactification of the Teichmüller space (see Fathi-Laudenbach-Poénaru [2]) or the argument in Otal [5], we have  $\text{length}_{\Sigma_i^-}(\lambda_i^-) \rightarrow \infty$ . Since  $\lambda_j^-$  is realised on  $\Sigma^-$ , its length on  $\Sigma_i^-$  with respect to  $p_i$  is equal to that in  $M_i$ . Therefore we have  $\text{length}_{M_i}(\Phi_i(\lambda_i^-)) \rightarrow \infty$ . On the other hand, by the continuity of the length function (Brock [1]), we have

$$\lim \text{length}_{M_i}(\Phi_i(\lambda_i^-)) = \text{length}_N(\Psi(\lambda_\infty^-))$$

and the right hand side is finite. This is a contradiction, and we have completed the proof of the first case.

Now let us turn to the second case. Let  $\lambda_0$  be the component shared by  $|\lambda_\infty^+|$  and  $|\lambda_\infty^-|$ , which is not a simple closed curve.

Using the technique of interpolating pleated surfaces due to Thurston, we prove the following.

**Proposition 3.** *We can take a constant  $L > 0$  for which the following holds for large  $i$ . There is  $t_i \in [0, 1]$  such that  $H_i(S(\mu_0), t_i)$  is homotopic*

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to  $f_i|S(\mu_0)$  by a homotopy staying within the distance  $L$  from  $f_i(S(\mu_0))$  which keeps the frontier inside the Margulis tubes all the time.

Then the pleated surface  $g_i|S(\mu_0)$  converges to a pleated surface  $g_\infty : S(\mu_0) \rightarrow M_\infty$  homotopic to  $f_\infty$  since the homotopy between  $g_i|S(\mu_0)$  and  $f_i|S(\mu_0)$  has bounded diameter and converges to a homotopy between  $g_\infty$  and  $f_\infty|S(\mu_0)$ . The limit pleated surface  $g_\infty$  realises the limit of the measured laminations  $\alpha_i(t_i)|S(\mu_0)$ . By taking a subsequence we can assume that  $\alpha(t_i)$  converges to a projective lamination on  $\alpha([0, 1])$ , which must have the same support as  $\mu_0$  if it is restricted in  $S(\mu_0)$ . Therefore the limit pleated surface realises  $\mu_0$ . Since  $f_\infty$  is lifted to  $f' : S \rightarrow N$ , the pleated surface  $g_\infty$  is also lifted to a pleated surface, which also realises  $\mu_0$ . This contradicts the fact that  $\mu_0$  represents an ending lamination. Thus we have completed the proof of Theorem 1 in this case.

The third case is the most difficult. We need to make an eclectic approach considering Hausdorff limits of the bending loci. The key steps are as follows.

**Lemma 4.** *Let  $\ell$  be a minimal component of  $\ell_\infty^-$  or  $\ell_\infty^+$ . Then  $\ell$  does not intersect a component of  $\mu$  transversely.*

**Lemma 5.** *Suppose that the Hausdorff limits  $\ell_\infty^\mp$  of  $|\lambda_i^\mp|$  contain a common component which coincides with the support a component  $\mu_0$  of  $\mu^\mp$ . Then there is an arc  $\alpha_i : [0, 1] \rightarrow \mathcal{PML}(S)$  connecting  $[\lambda_i^-]$  with  $[\lambda_i^+]$  converging uniformly to an arc  $\alpha_\infty$  such that for any sequence  $\{t_k\}$  in  $[0, 1]$  and monotone increasing  $\{i_k\}$  for which  $|\alpha_{i_k}(t_k)|$  converges in the Hausdorff topology, the limit contains a minimal component which coincides with  $|\mu_0|$  except for the case when  $t_k = 1/4i_k$  or  $t_k = 1 - 1/4i_k$  for all large  $k$ , in which case we have  $[\alpha_{i_k}(t_k)] = [\lambda_\infty^+]$  or  $[\lambda_\infty^-]$ .*

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